# 复变期中小测

邓嘉驹

2022 年 12 月 30 日

1. 该函数奇点满足 $\cos z = \dfrac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2} = -2$, 整理得

$$(\mathrm{e}^{\mathrm{i}z})^2 + 4\mathrm{e}^{\mathrm{i}z} + 1 = 0,$$

从中解得 $\mathrm{e}^{\mathrm{i}z} = -2 \pm \sqrt{3} < 0$，因此其全部奇点为

$$\begin{aligned} z_{k1} &= -\mathrm{i}\mathrm{Ln}\left(-2-\sqrt{3}\right) = -\mathrm{i}\left[\ln\left(2+\sqrt{3}\right) + \mathrm{i}\pi + 2k_1\pi\mathrm{i}\right] \\ &= (2k_1+1)\pi - \mathrm{i}\ln\left(2+\sqrt{3}\right) \quad (k_1 \in \mathbb{Z}), \\ z_{k2} &= -\mathrm{i}\mathrm{Ln}\left(-2+\sqrt{3}\right) = -\mathrm{i}\left[\ln\left(2-\sqrt{3}\right) + \mathrm{i}\pi + 2k_2\pi\mathrm{i}\right] \\ &= (2k_2+1)\pi - \mathrm{i}\ln\left(2-\sqrt{3}\right) \quad (k_2 \in \mathbb{Z}). \end{aligned}$$

2. (1) 经计算，

$$\mathrm{i}^{\sqrt{3}} = \exp\left(\sqrt{3}\mathrm{Ln}\ \mathrm{i}\right) = \exp\left[\sqrt{3}(0 + \mathrm{i}\mathrm{Arg}\ \mathrm{i})\right] = \exp\left[\frac{\sqrt{3}}{2}(4k+1)\pi\mathrm{i}\right] \quad (k \in \mathbb{Z}).$$

(2) 对 $\forall x \in \mathbb{R}$，由于

$$\sin(\mathrm{i}x) = \frac{\mathrm{e}^{-x} - \mathrm{e}^{x}}{2\mathrm{i}} = \mathrm{i}\sinh x, \quad \cos(\mathrm{i}x) = \frac{\mathrm{e}^{-x} + \mathrm{e}^{x}}{2} = \cosh x,$$

因此

$$\sin(1+2\mathrm{i}) = \sin 1\cos(2\mathrm{i}) + \cos 1\sin(2\mathrm{i}) = \sin 1\cosh 2 + \mathrm{i}\cos 1\sinh 2.$$

3. 记 $f(z) = \dfrac{1}{z(z-2\mathrm{i})(2z+2)^2}$. 在闭路 $C$ 所包围的区域内，$f(z)$ 具有 1 级极点 $z_0 = 0, z_2 = 2\mathrm{i}$ 与 2 级极点 $z_1 = -1$，由留数定理，

$$\begin{aligned} I &= \oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=0}^{2}\mathrm{Res}[f(z), z_k] \\ &= 2\pi\mathrm{i}\left\{\frac{1}{(-2\mathrm{i})\cdot 2^2} + \lim_{z\to z_1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2\cdot\frac{1}{z(z-2\mathrm{i})(2z+2)^2}\right] + \frac{1}{2\mathrm{i}\cdot(2+4\mathrm{i})^2}\right\} \\ &= 2\pi\mathrm{i}\cdot\left[-\frac{1}{8\mathrm{i}} + \frac{1}{8\mathrm{i}}\left(1 - \frac{1}{(1+2\mathrm{i})^2}\right) + \frac{1}{8\mathrm{i}(1+2\mathrm{i})^2}\right] = 0. \end{aligned}$$

4. 记 $f(z)=z\bar{z}=x^2+y^2$，其中 $x=\mathrm{Re}(z)\in\mathbb{R}, y=\mathrm{Im}(z)\in\mathbb{R}$，可见 $f(z)$ 在复平面上不解析，不能使用柯西积分定理. 当 $z$ 在圆 $|z|=1$ 上时，$f(z)=|z|^2=1$，记 $z=\mathrm{e}^{\mathrm{i}\theta}$，则由定义，

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &= \int_{-\mathrm{i}}^{\mathrm{i}} f(z)\Big|_{x=\mathrm{Re}(z)\equiv 0}\,\mathrm{d}z + \int_{\substack{|z|=1,\\ \pi/2<\theta<3\pi/2}} f(z)\,\mathrm{d}z \\ &= \mathrm{i}\int_{-1}^{1} y^2\,\mathrm{d}y + \int_{\pi/2}^{3\pi/2} \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \frac{2}{3}\mathrm{i}-2\mathrm{i}=-\frac{4}{3}\mathrm{i}.\end{aligned}$$

5. 记 $g(x,y)=x^2-y^2+xy+3y$，因此

$$\frac{\partial^2 g}{\partial x^2}=\frac{\partial}{\partial x}(2x+y)=2,\quad \frac{\partial^2 g}{\partial y^2}=\frac{\partial}{\partial y}(x-2y+3)=-2,$$

可得 $\Delta g=\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}=0$. 该式对 $\forall (x,y)\in\mathbb{R}^2$ 都成立，因此 $g(x,y)$ 为全平面的调和函数. 设 $f(z)=u(x,y)+\mathrm{i}v(x,y)$，其中 $z=x+\mathrm{i}y$ $(x,y\in\mathbb{R})$，那么

$$f'(z)=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-\mathrm{i}\frac{\partial v}{\partial y}.$$

由题意，

$$x^2-y^2+3y=\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y},$$

那么利用 $\Delta v=0$ 可得

$$2\frac{\partial^2 u}{\partial x\partial y}=(-2y+3)+2x,\quad \left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)u=2x-(-2y+3),\quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u=0,$$

即有

$$\frac{\partial^2 u}{\partial x^2}=x+y-\frac{3}{2},\quad \frac{\partial^2 u}{\partial x\partial y}=x-y+\frac{3}{2},\quad \frac{\partial^2 u}{\partial y^2}=-x-y+\frac{3}{2},$$

由题意，

$$f'(0)=\frac{\partial u}{\partial x}\Big|_{(0,0)}-\mathrm{i}\,\frac{\partial u}{\partial y}\Big|_{(0,0)}=0,\quad f(0)=u(0,0)+\mathrm{i}v(0,0)=\mathrm{i},$$

即有

$$u(0,0)=0,\quad v(0,0)=1;\quad \frac{\partial u}{\partial x}\Big|_{(0,0)}=\frac{\partial u}{\partial y}\Big|_{(0,0)}=0,$$

因此

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial x}\Big|_{(0,0)}+\int_0^x \frac{\partial^2 u}{\partial x^2}\Big|_{y=0}\,\mathrm{d}x+\int_0^y \frac{\partial^2 u}{\partial x\partial y}\Big|_{x}\,\mathrm{d}y=\frac{1}{2}(x^2+2xy-y^2)-\frac{3}{2}(x-y),\\ \frac{\partial u}{\partial y}&=\frac{\partial u}{\partial y}\Big|_{(0,0)}+\int_0^x \frac{\partial^2 u}{\partial x\partial y}\Big|_{y=0}\,\mathrm{d}x+\int_0^y \frac{\partial^2 u}{\partial y^2}\Big|_{x}\,\mathrm{d}y=\frac{1}{2}(x^2-2xy-y^2)+\frac{3}{2}(x+y),\end{aligned}$$

将其代入原方程组，并重复一次操作，可得

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x} - (x^2 - y^2 + 3y) = -\frac{1}{2}(x^2 - 2xy - y^2) - \frac{3}{2}(x + y),\\
\frac{\partial v}{\partial y} &= (x^2 - y^2 + 3y) - \frac{\partial u}{\partial y} = \frac{1}{2}(x^2 + 2xy - y^2) - \frac{3}{2}(x - y),\\
u(x,y) &= u(0,0) + \int_0^x \left.\frac{\partial u}{\partial x}\right|_{y=0} \mathrm{d}x + \int_0^y \left.\frac{\partial u}{\partial y}\right|_{x} \mathrm{d}y\\
&= \frac{1}{6}(x^3 + 3x^2y - 3xy^2 - y^3) - \frac{3}{4}(x^2 - 2xy - y^2),\\
v(x,y) &= v(0,0) + \int_0^x \left.\frac{\partial v}{\partial x}\right|_{y=0} \mathrm{d}x + \int_0^y \left.\frac{\partial v}{\partial y}\right|_{x} \mathrm{d}y\\
&= -\frac{1}{6}(x^3 - 3x^2y - 3xy^2 + y^3) - \frac{3}{4}(x^2 + 2xy - y^2) + 1.
\end{aligned}$$

综上分析，$f(z)$ 表达式为

$$f(z) = \frac{1}{6}(1 - \mathrm{i})z^3 - \frac{3}{4}(1 + \mathrm{i})z^2 + \mathrm{i}.$$

**简单解法**　设 $f'(z) = u(x,y) + \mathrm{i}v(x,y)$，由题意，$u - v = x^2 - y^2 + 3y$，因此

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -2y + 3,$$

由 C-R 方程，$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$，$\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$，代入消去 $v$ 可得

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2x, \quad \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = -2y + 3,$$

因此

$$\frac{\partial u}{\partial x} = x + y - \frac{3}{2}, \quad \frac{\partial u}{\partial y} = x - y + \frac{3}{2},$$

选取折线路径：$(0,0) \to (x,0) \to (x,y)$，可得

$$\begin{aligned}
u(x,y) &= u(0,0) + \int_0^x \left(u - \frac{3}{2}\right) \mathrm{d}u + \int_0^y \left(x - v + \frac{3}{2}\right) \mathrm{d}v\\
&= \frac{1}{2}x^2 - \frac{3}{2}x + xy - \frac{1}{2}y^2 + \frac{3}{2}y + C_1,\\
v(x,y) &= u(x,y) - (x^2 - y^2 + 3y) = -\frac{1}{2}x^2 - \frac{3}{2}x + xy + \frac{1}{2}y^2 - \frac{3}{2}y + C_1,
\end{aligned}$$

由于 $f'(0) = 0$，故 $C_1 = 0$，则

$$\begin{aligned}
f'(z) = u(x,y) + \mathrm{i}v(x,y) &= \frac{1}{2}(1 - \mathrm{i})x^2 + (1 + \mathrm{i})xy - \frac{1}{2}(1 - \mathrm{i})y^2 - \frac{3}{2}(1 + \mathrm{i})x + \frac{3}{2}(1 - \mathrm{i})y\\
&= \frac{1}{2}(1 - \mathrm{i})(x + \mathrm{i}y)^2 - \frac{3}{2}(1 + \mathrm{i})(x + \mathrm{i}y) = \frac{1}{2}(1 - \mathrm{i})z^2 - \frac{3}{2}(1 + \mathrm{i})z,
\end{aligned}$$

代入 $f(0) = \mathrm{i}$，积分得最终结果

$$f(z) = \frac{1}{6}(1 - \mathrm{i})z^3 - \frac{3}{4}(1 + \mathrm{i})z^2 + \mathrm{i}.$$

6. 记 $z = x + \mathrm{i}y$ $(x, y \in \mathbb{R})$，那么 $|z| < 1 - \mathrm{Re}\ z$ 可表示为 $\sqrt{x^2 + y^2} < 1 - x$，即区域 $D : 2x < 1 - y^2$. 曲线 $C : 2x = 1 - y^2$ 在 $\mathbb{R}^2$ 平面内为一以 $x$ 轴（在复平面内为实轴）为对称轴、焦准距 $p = 1$、焦点位于原点的抛物线，因此 $D$ 在复平面内表示位于 $C$ 外包含焦点的区域（不包含曲线 $C$）.

