第三周作业

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更新：March 24, 2025

1 Exercise 9.2

1. (9.2.2)
   (a). $z_x = 3^{-y/x}\ln 3\cdot y/x^2,\ z_y = 3^{-y/x}\ln 3\cdot(-1/x)$.
   (b). 注意到
   $$z = \arctan\frac{x+y}{x-y} = \arctan\frac{1+(y/x)}{1-(y/x)} = \arctan\frac{y}{x}+\frac{\pi}{4}, \tag{1.1}$$
   因此 $z_x = -y/(x^2+y^2),\ z_y = x/(x^2+y^2)$.
   (c). $u_x = y_z\cdot x^{y^z-1},\ u_y = x^{y^z}\ln x\cdot zy^{z-1},\ u_z = x^{y^z}\ln x\cdot y^z\ln y$.
   (d). $u_x = \mathrm{e}^{-z}+1/(x+\ln y),\ u_y = 1/[y(x+\ln y)],\ u_z = -x\mathrm{e}^{-z}+1$.
2. (9.2.3) $f_x = 2\sin\left(x^2y\right)/x,\ f_y = \sin\left(x^2y\right)/y$.
3. (9.2.4) 由题意，
   $$f_x(0,0) = \lim_{x\to0}\frac{f(x,0)-f(0,0)}{x} = 0,$$
   $$f_y(0,0) = \lim_{y\to0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to0}\sin\frac{1}{y^2}\text{振荡发散}.$$
4. (9.2.6) 参数方程为 $\begin{cases} y = 4, \\ z = x^2/4+4, \end{cases}$ 因此切向量
   $$\boldsymbol{\tau} = \left(1,0,\frac{x}{2}\right)\Big|_{x=2} = (1,0,1), \tag{1.2}$$
   从而 $\boldsymbol{\tau}$ 与 Ox 轴的夹角为 $\pi/4$.
5. (9.2.8) 由题意，
   $$u_t = -\frac{1}{2t^{3/2}}\mathrm{e}^{-x^2/4t}+\frac{x^2}{4t^{5/2}}\mathrm{e}^{-x^2/4t},$$
   $$u_x = -\frac{x}{2t^{3/2}}\mathrm{e}^{-x^2/4t},$$
   $$u_{xx} = -\frac{1}{2t^{3/2}}\mathrm{e}^{-x^2/4t}+\frac{x}{4t^{5/2}}\mathrm{e}^{-x^2/4t}.$$

6. (9.2.11) 直角坐标系下计算略，以下另外给两种解法：

   (a). Method 1: 采用球坐标下的 Laplacian（背公式法），即

$$\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 f. \tag{1.3}$$

   由于 $f(\boldsymbol{r}) = f(r)$，即函数只与位矢 $\boldsymbol{r}$ 的长度 r 有关，故仅取第一项，容易证明.

   (b). Method 2: 采用不依赖于坐标系的 Leibniz 公式方法（不背公式法，推荐）. 容易看出 $\nabla r = \hat{\boldsymbol{r}}$, $\nabla\cdot\boldsymbol{r} = 3$，因此

$$\nabla^2 r = \nabla\cdot\nabla r = \nabla\cdot\left(\frac{\boldsymbol{r}}{r}\right) = \frac{1}{r}\nabla\cdot\boldsymbol{r} + \boldsymbol{r}\cdot\nabla(\frac{1}{r}) = \frac{3-1}{r} = \frac{2}{r}. \tag{1.4}$$

   类似地，另外两问也容易证明：

$$\nabla^2\ln r = \nabla\cdot\frac{\hat{\boldsymbol{r}}}{r} = \nabla\cdot\frac{\boldsymbol{r}}{r^2} = \frac{3-2}{r^2} = \frac{1}{r^2}, \tag{1.5}$$

$$\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\left(-\frac{\boldsymbol{r}}{r^3}\right) = 0. \tag{1.6}$$

7. (9.2.13)

   (a). $\mathrm{d}z = (x\,\mathrm{d}y - y\,\mathrm{d}x)/(x^2+y^2)$.

   (b). $\mathrm{d}z = 4x(x^2-2y^2)\,\mathrm{d}x + 4y(y^2-2x^2)\,\mathrm{d}y$，因此 $\mathrm{d}z\big|_{(0,0)} = 0$, $\mathrm{d}z\big|_{(1,1)} = -4(\mathrm{d}x+\mathrm{d}y)$.

8. (9.2.15) 注意到 $f(0,0)=0$，且 $f_x(0,0) = \frac{\mathrm{d}f(x,0)}{\mathrm{d}x}\Big|_{x=0} = 0 = f_y(0,0)$，故

$$\lim_{\rho\to0^+}\frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\rho} = \lim_{\rho\to0^+}\sqrt{|\cos\theta(\rho)\sin\theta(\rho)|}, \tag{1.7}$$

   可见当 $f(x,y)$ 沿任一过极坐标原点的曲线 $\rho=\rho(\theta)$ 趋于原点时，上述极限不存在，故 $f(x,y)$ 在原点不可微.

9. (9.2.16) 先证明连续性：

$$\lim_{\rho\to0^+} f(x,y) \xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} \lim_{\rho\to0^+}\rho\cos^2\theta(\rho)\sin\theta(\rho) = 0, \tag{1.8}$$

   其中已利用夹逼准则 $0<|f(x,y)|\leqslant\rho$. 再证明不可微：由定义容易证明[1] $f_x(0,0) = f_y(0,0) = 0$，因此

$$\lim_{\rho\to0^+}\frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\rho} = \lim_{\rho\to0^+}\cos^2\theta(\rho)\sin\theta(\rho). \tag{1.9}$$

   例如当 $f(x,y)$ 从不同方向沿射线趋于原点时，上述极限值不同，故该极限不存在，从而 $f(x,y)$ 在原点不可微.

10. (9.2.20)

    (a). $u'(t) = f_x\cos t - f_y\sin t + f_z\mathrm{e}^t$.

    (b). $u_x = 2xf_1 + y\mathrm{e}^{xy}f_2$ 易求，二阶偏导给同学们一个防止漏项的求解示范：

$$\begin{aligned} u_{xy} &= 2x(-2yf_{11} + x\mathrm{e}^{xy}f_{12}) + (1+xy)\mathrm{e}^{xy}\cdot f_2 + y\mathrm{e}^{xy}(-2yf_{21} + x\mathrm{e}^{xy}f_{22}) \\ &= -4xyf_{11} + 2(x^2-y^2)\mathrm{e}^{xy}f_{12} + xy\mathrm{e}^{2xy}f_{22} + (1+xy)\mathrm{e}^{xy}f_2. \end{aligned} \tag{1.10}$$

[1] 助教偷懒，考试不能这么写，毕竟算分的

(c). $u_x = f_1 + 2xf_2$, $u_{xx} = f_{11} + 4xf_{12} + 4x^2 f_{22} + 2f_2$, $u_{xy} = f_{11} + 2(x+y)f_{12} + 4xyf_{22}$.

11. (9.2.25) $u_x = f'(t)(y\varphi_1 + \varphi_2)$, $u_y = f'(t)(x\varphi_1 + \varphi_2)$, $u_{xy} = f''(t)(y\varphi_1 + \varphi_2)(x\varphi_1 + \varphi_2) + f'(t)[xy\varphi_{11} + (x+y)\varphi_{12} + \varphi_{22} + \varphi_1]$.
12. (9.2.26) $xz_x - yz_y = x \cdot yf' - y \cdot xf' = 0$.
13. (9.2.27) $xz_x + y^2 z_y = x \cdot (f'/x) + y^2 \cdot f' \cdot (-1/y^2) = 0$.
14. (9.2.28) Trivial，懒得写了.
15. (9.2.32) 对于有显示解的变量代换方程组，千万不要上来就剥蒜！ 这里还是给两种不同解法，适合不同宝宝体质（雾）.

(a). Method 1: 简单易懂的方法——反解方程组得到 x, y 关于 u, v 的表达式，这样只用算一次二阶偏导（原本要算三次），即 $\begin{cases} x = (au + 2v)/(a+2), \\ y = (v-u)/(a+2), \end{cases}$ 从而

$$z_{uv} = \partial_v\left(\frac{az_x - z_y}{a+2}\right) = \frac{2az_{xx} + (a-2)z_{xy} - z_{yy}}{(a+2)^2} = 0. \tag{1.11}$$

对比 z_{xx} 和 z_{yy} 的系数比，解得 $a = 3$.

(b). Method 2: PDE 降维打击（视情况可能习题课讲点），这是二阶线性齐次 PDE，因此解以下方程：

$$6(\mathrm{d}y)^2 - (\mathrm{d}x)(\mathrm{d}y) - (\mathrm{d}x)^2 = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \text{ or } -\frac{1}{3}, \tag{1.12}$$

即 $x - 2y = C_1$ 和 $x + 3y = C_2$. 因此对于此双曲型方程，作变量代换 $\xi = x - 2y$, $\eta = x + 3y$，必然可将其简化为 $z_{\xi\eta} = 0$. 对比系数可知，$a = 3$.

2 Exercise 9.3

1. (9.3.1 (1)) 由题意，记 $F(x, y) = x^2 + xy + y^2 - 7$，则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{2x+y}{2y+x}, \tag{2.1}$$

可见在 $x, y > 0$ 的区域内均有 $\mathrm{d}y/\mathrm{d}x < 0$，该方程在点 $(2,1)$ 附近有唯一解，且

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(2,1)} = -\frac{5}{4}, \quad \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{(2,1)} = \left.\frac{3(xy'-y)}{(2y+x)^2}\right|_{(2,1)} = -\frac{21}{32}. \tag{2.2}$$

注 计算隐函数在定点的高阶导时，可将其中已计算出的低阶导数值整体代入，不必再代入其代数表达式.

2. (9.3.2)

(a). 按汪爷爷提到的书写规范，记 $F(x, y) = \ln(x^2 + y^2)/2 - \arctan(y/x)$，则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = \frac{x+y}{x-y}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2(xy'-y)}{(x-y)^2} = \frac{2(x^2+y^2)}{(x-y)^3}. \tag{2.3}$$

(b). 注意到 z_x, z_y 可完整构成全微分 $\mathrm{d}z$，故对原方程取微分，避免两次计算微分（偏导数）：

$$\frac{z\,\mathrm{d}x - x\,\mathrm{d}z}{z^2} = \frac{\mathrm{d}z}{z} - \frac{\mathrm{d}y}{y} \quad \Longrightarrow \quad \mathrm{d}z = \frac{z(y\,\mathrm{d}x + z\,\mathrm{d}y)}{y(x+z)}, \tag{2.4}$$

从中可得 $z_x = z/(x+z)$, $z_y = z^2/[y(x+z)]$.

3. (9.3.4 (1)) 由题意，

$$\sin 2x\,\mathrm{d}x + \sin 2y\,\mathrm{d}y + \sin 2z\,\mathrm{d}z = 0 \quad \Longrightarrow \quad \mathrm{d}z = -\frac{\sin 2x\,\mathrm{d}x + \sin 2y\,\mathrm{d}y}{\sin 2z}. \tag{2.5}$$